Nonlinear analysis of Reissner's plate by the variational approaches and boundary element methods

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The purpose of the paper is twofold, first to suggest a general method, called the second variation–convex analysis mixed method, for analysis of nonlinear functionals. As an application of the approach, a strictly dual complementary variational principle of Reissner plates with the local form[1] and the existing criterion of variational solution have been studied. The study shows that the principle and the criterion[1] can be improved into a global form by using the novel approach. In contrast with existing methods (Jin[1] and Gao and Stang[2]), this method will be able to analyze general nonlinear problems,[1] rather than geometrically nonlinear ones.[2] Our second purpose is to present a new approach to the derivation of the exact boundary integral equation for the analysis of nonlinear Reissner plates and to the derivation of the criterion for the solution of the boundary integral equation. Subsequently, the boundary and the domain of the plate are discretized to solve the nonlinear problems. All unknown variables are at the boundary. Numerical results are presented to illustrate the method and demonstrate its effectiveness and accuracy.

Keywords: variational principle, second variation–convex analysis mixed method, nonlinear Reissner plate

Introduction

The study of the strictly dual complementary principle with its elegant and symmetric characteristics has received considerable attention. The representative methodology can be found in the work of Gao and Stang[2] for the global form of geometrically nonlinear problems with a convex analysis approach and in Jin's work[1] on the local form of nonlinear problems of Reissner plates using the second variation tool. By strictly dual complementarity we mean a pair of dual complementary variational functionals have the same criterion for the existence of variational solutions.

On the other hand, several researchers have investigated finite deformation behavior of plates, such as Kamiya et al.,[3] Tanaka,[4] and Qin[5] for thin plates and Lei et al.[6] for moderately thick plates. In the work reported by Lei et al.,[6] an integral model for analyzing finite deflection of an isotropic plate taking into account the transverse shear deformation is deduced by a weighted residual method. In the course of derivation, the nonlinear terms are treated as the pseudo-transverse distributed load, which means that the nonlinear terms are considered as the known external loads in the analysis.

In this study, we endeavor to develop an approach that unifies the aforementioned two methods of Jin[1] and Gao and Stang,[2] respectively. Our study indicates that both the strictly dual complementary principles and the existent criterion of variational solution with the local form[1] can be improved into a variant version with a refined global form. Obviously the approach includes the advantages of the previous two methods. We call this approach the second variation–convex analysis mixed method.

With regard to the boundary element method, we present a set of exact boundary integral equations of nonlinear Reissner plates. In contrast with previous work,[6] the nonlinear terms are treated as dependent on unknown displacements and stresses, rather than the pseudo-load. Furthermore, the integral equation is derived on the basis of variational method, not the weighted residual method. To make the derivation tractable, a modified variational function for the analysis of the geometrically nonlinear plate and the existent criterion of variational solution for the function are presented originally. Finally as an application of the proposed method, a number of numerical examples are given. The results are in good agreement with already existing solutions. The proposed method appears very promising.

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Variational principles

Consider an isotropic plate of uniform thickness h with mid-plane coordinates x and y. Indices i, j, and k have values in the range $\{1, 2\}$. The governing equations,[1] which include the effects of transverse shear deformation, are

$$\Omega:\quad \left.\begin{aligned} &N_{ij,j} = 0 \\ &M_{ij,j} - Q_i = 0 \\ &Q_{i,i} + N_{ij}w_{,ij} + q = 0 \end{aligned}\right] \tag{1}$$

$$\left.\begin{aligned} &\epsilon_{ij} = (u_{i,j} + u_{j,i} + w_{,i}w_{,j})/2 \\ &\theta_{ij} = (\phi_{i,j} + \phi_{j,i})/2 \\ &\Psi_i = \phi_i + w_{,i} \end{aligned}\right] \tag{2}$$

$$N_{ij} = \partial U_N/\partial\epsilon_{ij};\ M_{ij} = \partial U_M/\partial\theta_{ij};\ Q_i = \partial U_Q/\partial\Psi_i \tag{3}$$

$$\epsilon_{ij} = \partial V_N/\partial N_{ij};\ \theta_{ij} = \partial V_M/\partial M_{ij};\ \Psi_i = \partial V_Q/\partial Q_i \tag{4}$$

$$U + V = N_{ij}\epsilon_{ij} + M_{ij}\theta_{ij} + Q_i\Psi_i \tag{5}$$

and the boundary conditions are described by

$$\begin{aligned} &C_{N_n}: N_n = N_{ij}n_in_j = \bar{N}_n;\ C_{N_{ns}}: N_{ns} = N_{ij}n_is_j = \bar{N}_{ns}; \\ &C_{M_n}: M_n = M_{ij}n_in_j = \bar{M}_n;\ C_{M_{ns}}: M_{ns} = M_{ij}n_is_j = \bar{M}_{ns}; \\ &C_R:\ R_n = Q_in_i + N_nw_{,n} + N_{ns}w_{,s} = \bar{R}_n \end{aligned} \tag{6a–e}$$

$$\begin{aligned} &C_{u_n}: u_n = u_in_i = \bar{u}_n;\ C_{u_s}: u_s = u_is_i = \bar{u}_s; \\ &C_{\phi_n}: \phi_in_i = \bar{\phi}_n;\ C_{\phi_s}: \phi_s = \phi_is_i = \bar{\phi}_s; \qquad C_w: w = \bar{w} \\ &\quad (\partial\Omega = C_{u_n}\cup C_{N_n} = C_{u_s}\cup C_{N_{ns}} = C_{\phi_n}\cup C_{M_n} \\ &\quad = C_{\phi_s}\cup C_{M_{ns}} = C_w\cup C_R) \end{aligned} \tag{7a–e}$$

where (1) represents the equilibrium equations; (2) the geometric equations; (3), (4), and (5) are three equivalent forms of constitutive equations; (6) and (7) are the boundary conditions; and all unstated symbols are listed in the nomenclature section at the end of the paper.

For the boundary value problem (1)–(7), we derive, now, systematically a set of fundamental variational principles with three, two, or one field(s) of independent variables subject to variation:

1. Hu–Washizu principle: $\delta\Pi_3 = \delta\Gamma_3 = 0$

$$\begin{aligned} \Pi_3 = &\int_\Omega \{U - N_{ij}[\epsilon_{ij} - (u_{i,j} + u_{j,i} + w_{,i}w_{,j})/2] - M_{ij}[\theta_{ij} - (\phi_{i,j} + \phi_{j,i})/2] \\ &- Q_i[\Psi_i - (\phi_i + w_{,i})] - wq\}\, d\Omega + \int_{c_{u_n}} (\bar{u}_n - u_n)N_n\, dc \\ &+ \int_{c_{u_s}} (\bar{u}_s - u_s)N_{ns}\, dc + \int_{c_{\phi_n}} (\bar{\phi}_n - \phi_n)M_n\, dc + \int_{c_{\phi_s}} (\bar{\phi}_s - \phi_s)M_{ns}\, dc + \int_{c_w} (\bar{w} - w)R_n\, dc \\ &- \int_{c_{N_s}} \bar{N}_nu_n\, dc - \int_{c_{N_{ns}}} \bar{N}_{ns}u_s\, dc - \int_{c_{M_n}} \bar{M}_n\phi_n\, dc - \int_{c_{M_{ns}}} \bar{M}_{ns}\phi_s\, dc - \int_{c_R} \bar{R}_nw\, dc \end{aligned} \tag{8}$$

$$\begin{aligned} \Gamma_3 = &\int_\Omega \{U - N_{ij,j}u_i - (M_{ij,j} - Q_i)\phi_i - (Q_{i,j} + N_{ij}w_{,ij} + q)w \\ &- N_{ij}\epsilon_{ij} - (N_{ij}w_{,i}w_{,j})/2 - M_{ij}\theta_{ij} - Q_i\Psi_i\}\, d\Omega + \int_{c_{N_n}} (N_n - \bar{N}_n)u_n\, dc \\ &+ \int_{c_{N_{ns}}} (N_{ns} - \bar{N}_{ns})u_s\, dc + \int_{c_{M_n}} (M_n - \bar{M}_n)\phi_n\, dc + \int_{c_{M_{ns}}} (M_{ns} - \bar{M}_{ns})\phi_s\, dc \\ &+ \int_{c_R} (R_n - \bar{R}_n)w\, dc + \int_{c_{u_n}} N_n\bar{u}_n\, dc + \int_{c_{u_s}} N_{ns}\bar{u}_s\, dc + \int_{c_{\phi_n}} M_n\bar{\phi}_n\, dc \\ &+ \int_{c_{\phi_s}} M_{ns}\bar{\phi}_s\, dc + \int_{c_w} R_n\bar{w}\, dc \end{aligned} \tag{9}$$

$$\Pi_3 = \Gamma_3 \tag{10}$$

with the stationary conditions (1), (2), (3), (6), and (7).

2. Hellinger–Reissner principle: $\delta\Pi_2 = \delta\Gamma_2 = 0$

$$\begin{aligned}\Pi_2 = \int_\Omega \{&-V + N_{ij}(u_{i,j} + u_{j,i} + w_{,i}w_{,j})/2 + M_{ij}(\phi_{i,j} + \phi_{j,i})/2 \\ &+ Q_i(\phi_i + w_{,i}) - wq\}\,d\Omega + \int_{c_{u_n}} (\bar{u}_n - u_n)N_n\,dc + \int_{c_{u_s}} (\bar{u}_s - u_s)N_{ns}\,dc \\ &+ \int_{c_{\phi_n}} (\bar{\phi}_n - \phi_n)M_n\,dc + \int_{c_{\phi_s}} (\bar{\phi}_s - \phi_s)M_{ns}\,dc + \int_{c_w} (\bar{w} - w)R_n\,dc \\ &- \int_{c_{N_n}} \bar{N}_n u_n\,dc - \int_{c_{N_s}} \bar{N}_{ns} u_s\,dc - \int_{c_{M_{ns}}} \bar{M}_{ns}\phi_s\,dc - \int_{c_R} \bar{R}_n w\,dc - \int_{c_{M_n}} \bar{M}_n\phi_n\,dc\end{aligned} \tag{11}$$

$$\begin{aligned}\Gamma_2 = \int_\Omega &\left\{-V - \frac{1}{2}N_{ij}w_{,i}w_{,j} - N_{ij,j}u_i - (M_{ij,j} - Q_i)\phi_i - (Q_{i,i} + N_{ij}w_{,ij} + q)w\right\}d\Omega \\ &+ \int_{c_{N_n}} (N_n - \bar{N}_n)u_n\,dc + \int_{c_{N_{ns}}} (N_{ns} - \bar{N}_{ns})u_s\,dc + \int_{c_{M_n}} (M_n - \bar{M}_n)\phi_n\,dc \\ &+ \int_{c_{M_{ns}}} (M_{ns} - \bar{M}_{ns})\phi_s\,dc + \int_{c_R} (R_n - \bar{R}_n)w\,dc + \int_{c_{u_n}} N_n\bar{u}_n\,dc \\ &+ \int_{c_{u_s}} N_{ns}\bar{u}_s\,dc + \int_{c_{\phi_n}} M_n\bar{\phi}_n\,dc + \int_{c_{\phi_s}} M_{ns}\bar{\phi}_s\,dc + \int_{c_w} R_n\bar{w}\,dc\end{aligned} \tag{12}$$

$$\Pi_2 = \Gamma_2 \tag{13}$$

where the stationary conditions are (1), (4), (6), and (7).

3. The generalized principle of total potential energy $\delta\Pi_1 = 0$

$$\Pi_1 = \int_\Omega (U - wq)\,d\Omega - \int_{c_{N_n}} \bar{N}_n u_n\,dc - \int_{c_{N_{ns}}} \bar{N}_{ns} u_s\,dc - \int_{c_{M_n}} \bar{M}_n\phi_n\,dc - \int_{c_{M_w}} \bar{M}_{ns}\phi_s\,dc - \int_{c_R} \bar{R}_n w\,dc \tag{14}$$

with stationary conditions (1) and (6).

4. The generalized principle of complementary energy $\delta\Gamma_1 = 0$

$$\Gamma_1 = \int_\Omega \left(V - \frac{1}{2}N_{ij}w_{,i}w_{,j}\right)d\Omega - \int_{c_{u_n}} N_n\bar{u}_n\,dc - \int_{c_{u_s}} N_{ns}\bar{u}_s\,dc - \int_{c_{\phi_n}} M_n\bar{\phi}_n\,dc - \int_{c_{\phi_s}} M_{ns}\bar{\phi}_s\,dc - \int_{c_w} R_n\bar{w}\,dc \tag{15}$$

in which the stationary conditions are (7).

All the proofs of the principles can be found in theorem 1.

Theorem 1

If inequality

$$\int_\Omega \frac{1}{2}N_{ij}\delta w_{,i}\delta w_{,j}\,d\Omega \geq 0 \quad \forall w \in \mathcal{U} \tag{16}$$

holds, we see

$$\inf(\Pi_1) = \Pi_{10} = \Pi_{30} = \Pi_{20} = -\Gamma_{10} = \sup(-\Gamma_1) \tag{17}$$

where $\mathcal{U}$ represents a kinematic admissible space, Π_{io} means the stationary value of Π_i at $\{u\}_o$, where $\{u\}_o$ is the exact solution of the boundary value problem (1)–(7). Moreover, if $\mathcal{U}$ is a bounded subset of a reflexive Banach space, expression (17) has at least one solution. Furthermore, the solution is unique if the inequality (16) holds strictly.

Proof: From the first, we prove that the solution of the boundary value problem (1)–(7) is the stationary conditions of the aforementioned functions. In doing this, taking variation of Γ_1, we have

$$\begin{aligned}\delta\Gamma_1 \underline{\underline{(1)(2)(4)(6)}} &\int_{c_{u_n}} (u_n - \bar{u}_n)\delta N_n\,dc \\ &+ \int_{c_{u_s}} (u_s - \bar{u}_s)\delta N_{ns}\,dc + \int_{c_{\phi_n}} (\phi_n - \bar{\phi}_n)\delta M_n\,dc \\ &+ \int_{c_{\phi_s}} (\phi_s - \bar{\phi}_s)\delta M_{ns}\,dc + \int_{c_w} (w - \bar{w})\delta R_n\,dc\end{aligned} \tag{18a}$$

$$\delta\Gamma_1 \underline{\underline{(1)(2)(4)(6)}} = 0 \rightarrow (7) \tag{18b}$$

where constrained equality $\underline{\underline{(1)(2)(4)(6)}}$ represents that (1, 2, 4, 6) are satisfied, a priori. In the same manner, we obtain

$$\delta\Pi_3 = \delta\Gamma_3 = 0 \rightarrow (1), (2), (3), (6), \text{and } (7) \quad (19)$$

$$\delta\Pi_2 = \delta\Gamma_2 = 0 \rightarrow (1), (4), (6), \text{and } (7) \quad (20)$$

$$\delta\Pi_1 \underset{=}{\underline{(2)(3)(7)}} 0 \rightarrow (1) \text{ and } (6) \quad (21)$$

It can be easily verified that $\Pi_{10} = \Pi_{30} = \Pi_{20} = -\Gamma_{10}$.

Finally, attention will be focused on proving the rest of theorem 1. To this end, taking variation of $\delta\Gamma_1$ (expression 18a), and using the constrained conditions (1), (2), (4), (6), one easily obtains (see Reference 1 for details).

$$\delta^2\Gamma_1 = \int_\Omega \left(\delta^2 U(\epsilon_{ij}, \theta_{ij}, \Psi_2) + \frac{1}{2} N_{ij}\delta w_{,i}\delta w_{,j}\right) d\Omega \quad (22)$$

It should be noted that the second variation $\delta^2 U$ in (22) is with respect to variable strain, not to the displacement. The second variation of Π_1, namely, $\delta^2\Pi_1$, can be derived similarly and we discover $\delta^2\Pi_1 = \delta^2\Gamma_1$, which means that Π_1 and Γ_1 are a pair of strictly complementary functions because U is a convex function of strain variables, but may not be the convex function of displacement variables. Therefore if inequality (16) holds, Π_1 and Γ_1 are two convex functions. So theorem 1 has been proved by means of the theory of convex analysis.

Obviously the important result obtained here involves results given by Gao and Stang[2] for a global form and Jin[1] for a local form, which shows also that the local form of strictly dual complementary principles and the existing criterion of variational solution with the local form[1] can be improved into a refined global form.

Boundary integral equations

In what follows we derive a set of exact boundary integral equations of nonlinear Reissner plates by way of modified variational principle. To start with, we construct a function Π_1' as below

$$\Pi_1' = \Pi_1 + \int_{c_{u_n}} (\bar{u}_n - u_n)N_n\,dc + \int_{c_{u_s}} (\bar{u}_s - u_s)N_{ns}\,dc + \int_{c_{\phi_n}} (\bar{\phi}_n - \phi_n)M_n\,dc + \int_{c_{\phi_s}} (\bar{\phi}_s - \phi_s)M_{ns}\,dc + \int_{c_w} (\bar{w} - w)R_n\,dc \quad (23)$$

in which we assume that (2) and (3) are identically satisfied.

Lemma 1. If inequality

$$\int_\Omega N_{ij}\delta w_{,i}\delta w_{,j}d\Omega - \int_{c_{u_n}} \delta u_n\delta N_n\,dc - \int_{c_{u_s}} \delta u_s\delta N_{ns}\,dc - \int_{c_{\phi_n}} \delta\phi_n\delta M_n\,dc - \int_{c_{\phi_s}} \delta\phi_s\delta M_{ns}\,dc - \int_{c_w} \delta w\delta R_n\,dc \geq 0 \quad (24)$$

holds in the neighborhood of the solution of (1)–(7), we have

$$\Pi_1' \geq \Pi_{10}' \quad (25)$$

where Π_{10}' represents the stationary value of Π_1' at the arguments $\{u\}_o$, and the equal sign holds if and only if the arguments of function Π_1' are at the critical point.

Proof: Taking a variation of Π_1' we see

$$\begin{aligned}\delta\Pi_1' = &\iint_\Omega \{-N_{ij,j}\delta u - (M_{ij,j} - Q)\delta\phi_i - (Q_{i,i} + N_{ij}w_{,ij} + q)\delta w\}\,d\Omega + \int_{c_{u_n}} (\bar{u}_n - u_n)\delta N_n\,dc + \int_{c_{u_s}} (\bar{u}_s - u_s)\delta N_{ns}\,dc \\ &+ \int_{c_w} (\bar{w} - w)\delta R_n\,dc + \int_{c_{\phi_n}} (\bar{\phi}_n - \phi_n)\delta M_n\,dc + \int_{c_{\phi_s}} (\bar{\phi}_s - \phi_s)\delta M_{ns}\,dc + \int_{c_{N_n}} (N_n - \bar{N}_n)\delta u_n\,dc \\ &+ \int_{c_{N_{ns}}} (N_{ns} - \bar{N}_{ns})\delta u_{ns}\,dc + \int_{c_R} (R_n - \bar{R}_n)\delta w\,dc + \int_{c_{M_n}} (M_n - \bar{M}_n)\delta\phi_n\,dc + \int_{c_{M_{ns}}} (M_{ns} - \bar{M}_{ns})\delta\phi_s\,dc \end{aligned} \quad (26)$$

$$\delta\Pi_1' \underset{=}{\underline{(2)(3)}} 0 \rightarrow (1), (6), \text{and } (7) \quad (27)$$

which means (1), (6), and (7) are the stationary conditions of Π_1'. Consequently calculating the second variation of Π_1', one obtains

$$\delta^2\Pi_1' = \int_\Omega \delta^2 U(\epsilon_{ij}, \theta_{ij}, \Psi)\,d\Omega + \text{the left-hand side of inequality (24)} \quad (28)$$

Because $\delta^2 U(\epsilon_{ij}, \theta_{ij}, \Psi) > 0$, $\delta^2\Pi_1'$ will be uniformly positive if (24) holds, and lemma 1 has been proved from the sufficient condition of local extreme of a function.[7]

Based on the function Π_1', we obtain

Theorem 2

The solutions of (1)–(7) satisfy the boundary integral equations:

$$\lambda x_k = -\int_{\Omega} \frac{1}{2} w_{,i} w_{,j} \overset{*}{N}{}^{(k)}_{ij} d\Omega + \int_{c_{N_n}} (\overset{*}{u}{}^{(k)}_n \bar{N}_n - u_n \overset{*}{N}{}^{(k)}_n)\, dc + \int_{c_{N_{ns}}} (\overset{*}{u}{}^{(k)}_s \bar{N}_{ns} - u_s \overset{*}{N}{}^{(k)}_{ns})\, dc + \int_{c_{u_n}} (\overset{*}{u}{}^{(k)}_n N_n - \bar{u}_n \overset{*}{N}{}^{(k)}_n)\, dc + \int_{c_{u_s}} (\overset{*}{u}{}^{(k)}_s N_{ns} - \bar{u}_s \overset{*}{N}{}^{(k)}_{ns})\, dc \quad (29)$$

$$\lambda x_m = \int_{\Omega} (\overset{*}{w}{}^{(m)} q - N_{ij} w_{,i} \overset{*}{w}{}^{(m)}_{,j})\, d\Omega + \int_{c_{M_n}} (\overset{*}{\phi}{}^{(m)}_n \bar{M}_n - \phi_n \overset{*}{M}{}^{(m)}_n)\, dc + \int_{c_{M_{ns}}} (\overset{*}{\phi}{}^{(m)}_s \bar{M}_{ns} - \phi_s \overset{*}{M}{}^{(m)}_{ns})\, dc + \int_{c_R} (\overset{*}{w}{}^{(m)} \bar{R}_n - w \overset{*}{Q}{}^{(m)}_n)\, dc + \int_{c_{\phi_n}} (\overset{*}{\phi}{}^{(m)}_n M_n - \bar{\phi}_n \overset{*}{M}{}^{(m)}_n)\, dc + \int_{c_{\phi_s}} (\overset{*}{\phi}{}^{(m)}_s M_{ns} - \bar{\phi}_s \overset{*}{M}{}^{(m)}_{ns})\, dc + \int_{c_w} (\overset{*}{w}{}^{(m)} R_n - \bar{w} \overset{*}{Q}{}^{(m)}_n)\, dc \quad (m = 3, 4, 5) \quad (30)$$

and the solution of (29) and (30) exists if inequality (24) holds, where $\{x\} = \{x_1\ x_2\ x_3\ x_4\ x_5\} = \{u_1\ u_2\ \phi_1\ \phi_2\ w\}$ is a displacement vector, λ a conventional boundary shape coefficient, and symbol $(*)^{(p)}$ represents the related function solution.[8,9]

Proof: Noting that the displacement vector $\{x\}$ in (26) is not constrained by the boundary condition (7), the quantity $\delta\{x\}$ can be arbitrarily assumed. Naturally let

$$\delta\{x\} = \epsilon\{\overset{*}{x}\}^{(p)} \qquad (p = 1, 2, \ldots, 5) \qquad (31)$$

where ϵ is an infinitesimal, the components $\overset{*}{x}{}^{(p)}_q(P, Q)$ of $\{\overset{*}{x}\}^{(p)}(P, Q)$ mean the in-plane displacements (for $q = 1$ and 2) or the rotations (for $q = 3$ and 4) or the deflection (for $q = 5$) at the field point Q of an infinite plate when a unit point force (for $p = 1$, 2, and 5) or a unit point couple (for $p = 3$ and 4) is applied at the source point P.

For the linear plate boundary expression, the fundamental solutions have been given in Refs. 8 and 9. Obviously the membranous equations (1a) and bending ones (1b) and (1c) are independent of each other in the case of linear elasticity, that is, $\overset{*}{x}{}^{(m)}_i = 0$ and $\overset{*}{x}{}^{(i)}_m = 0$. Thus (27) can be further transformed into (29) and (30) by using the property of fundamental vector $\{\overset{*}{x}\}^{(p)}$. Thus the solutions of (1)–(7) satisfy (29) and (30), while the existing condition for the solution of integral equations (29) and (30) can be obtained from lemma 1. This completes the proof.

Boundary element analysis

To obtain a solution of (29) and (30), the boundary $\partial\Omega$ and the solution domain Ω of a plate are, respectively, divided into a series of boundary elements and internal cells as in the usual boundary element model. After performing discretization by use of various kinds of boundary elements (e.g., constant element, linear element, or higher order element), (29) and (30) become two sets of linear algebraic equations including the variables N_n, N_{ns}, u_n, u_s, M_n, M_{ns}, ϕ_n, ϕ_s, Q_n, w. Of the 10 quantities, five need to be prescribed on the boundary points and the remaining five are to be determined. Because an incremental formulation may have a wider applicability to higher nonlinear problems it is necessary to express (29) and (30) in the incremental form. Denoting the incremental variable by the superimposed dot, (29) and (30) can be expressed in matrix form

$$[Q]\{\dot{N}\} + [S]\{\dot{u}\} = \{\dot{R}_1\} \qquad (32a)$$

$$[H]\{\dot{M}\} + [G]\{\dot{\phi}\} = \{\dot{R}_2\} \qquad (32b)$$

where $[Q]$, $[S]$, $[H]$, and $[G]$ denote the coefficient matrices that can be calculated in the usual way, whereas $\{\dot{N}\} = \{\dot{N}_n\ \dot{N}_{ns}\}$, $\{\dot{u}\} = \{\dot{u}_n\ \dot{u}_s\}$, $\{\dot{M}\} = \{\dot{M}_n\ \dot{M}_{ns}\ \dot{R}_n\}$, $\{\dot{\phi}\} = \{\dot{\phi}_n\ \dot{\phi}_s\ \dot{w}\}$, $\{\dot{R}_1\}$, and $\{\dot{R}_2\}$ contain the nonlinear inhomogeneous terms that can be deduced from equations (29) and (30). To compute the nonlinear terms, an iterative procedure is required. An efficient iterative scheme given by Qin and Huang[8] will be adopted in the boundary element analysis. For the sake of conciseness, we omit those straightforward.

Numerical examples

The performance of the present element model is illustrated by several benchmark problems. The examples include a geometrically nonlinear thin plate and two large deflection sandwich plates. In all the computations, one quarter of the problems are analyzed. To study the convergence properties of the present approach, 16 constant elements on the boundary and three meshes of internal cell (3 × 3, 4 × 4, 5 × 5) are used. The convergence tolerance is $\epsilon = 0.001$.

Example 1: An isotropic thin plate

A fully clamped, thin square plate is subjected to a uniform distributed loading q. The plate geometry and material properties are given as follows

$E = 2.1 \times 10^6$ kg/cm^2; $\nu = 0.316$; side length $2a = 762$ cm; plate thickness $t = 7.62$ cm and $Q = qa^4/Et^4$.

The numerical results describing the relationships between the maximum deflection w_c/t occurring at the center and the loading factor Q are shown in *Table 1*. Comparison is made with the known results.[10]

Example 2: Clamped square sandwich plate

The plate consists of two identical facings ($E = 0.74 \times 10^6$ kg/cm^2, $\nu = 0.3$, side length $2a = 127$ cm) that are $t = 0.381$ cm thick and an aluminum honeycomb core ($G_c = 0.35 \times 10^4$ kg/cm^2) that is $h = 2.54$ cm thick, which are subjected to a uniform transverse load q; the boundaries of the square plate are fully clamped so that the imposed displacement boundary conditions are $u_1 = u_2 = \phi_1 = \phi_2 = w = 0$ on the whole boundary.

Table 2 shows the results for central defection vs. load parameter $Q = 12a^3(1 - \nu^2)q/(th^2E)$, and comparison is made with results obtained by Schmit and Manforton.[11]

Example 3: Compressed sandwich plate

Consider a square ($2a = 59.7$ cm) simply supported sandwich plate with identical isotropic facings ($E = 0.668 \times 10^6$ kg/cm^2, $\nu = 0.3$, $t = 0.0533$ cm) and a 0.46 cm thick core ($G_c = 0.134 \times 10^4$ kg/cm^2), subjected to uniformly in-plane compress N_x at the boundary $x = \pm a$ (here the origin of the coordinate frame is laid at the center of the square plate). The displacement boundary conditions used are as follows

$$w = \phi_y = 0 \quad \text{on } x = \pm a$$

$$w = \phi_x = 0 \quad \text{on } y = \pm a$$

Assuming a symmetrical buckling pattern only a quadrant of the plate needs to be considered. The load N_x vs. central defections w_c is given in *Table 3* and comparison is made with the results reported in Ref. 11.

Table 1. Central deflection w_c/t of the plate.

Q	17.79	38.3	63.4	95	134.9
B 9 cells	0.2322	0.4603	0.6781	0.8809	1.0311
E 16	0.2351	0.4642	0.6812	0.8895	1.0432
M 25	0.2354	0.4647	0.6818	0.8910	1.0541
Ref. 10	0.2368	0.4699	0.6915	0.9029	1.1063

Table 2. Central deflection w_c/h of the sandwich plate.

Q	10	20	30	40
B 9 cells	0.720	1.311	1.692	1.917
E 16	0.715	1.289	1.659	1.858
M 25	0.713	1.282	1.647	1.841
Ref. 11*	0.70	1.26	1.62	1.82

* Values obtained from Fig. 4 on page 1458.

Table 3. Central deflection w_c (cm) of the compressed plate.

N_x	54.7	57.2	61.6	66
B 9 cells	0	0.432	0.770	0.968
E 16	0	0.445	0.789	0.991
M 25	0	0.450	0.790	1.005
Ref. 11*	0	0.457	0.813	1.02

* Values obtained from Fig. 5 on page 1458.

It can be seen from the tables that the results obtained by the present method agree well with the results reported in Refs. 10 and 11. The numerical results show little sensitivity to the varying internal mesh. In the course of computations convergence was achieved with between 28 and 42 iterations for each loading increment.

Concluding remarks

In scientific research, we often encounter a category of nonlinear problems that exceeds the limits of solid mechanics, such as gravitation theory.[12] However, the convex analysis method cannot be directly applied to these problems. To efficiently solve such a category of problems, we present a novel approach called the second variation–convex analysis mixed method and apply it to the analysis of nonlinear Reissner plates. We also present a general and effective method for establishing an exact nonlinear boundary integral equation and for deriving its solution. In fact, the method is based on a modified variational principle that is given in the paper. The approach shows that a boundary integral formulation can be exactly transformed from a modified variational functional. It also reveals the intrinsic relations between the variational principle and the boundary integral equation. The numerical examples show that the aforementioned boundary element model is very effective for nonlinear analysis of Reissner plates.

Nomenclature

C	$= 5\,Eh/12(1 + \nu)$ for homogeneous plate; $= G_c(h + t)$ for sandwich plate
C_w	a part of boundary $\partial\Omega$ of the solution domain Ω on which the deflection w is prescribed; C_R et al. can be defined similarly
D	$= Et^3/12(1 - \nu^2)$ for homogeneous plate; $= E(h + t)^2t/2(1 - \nu^2)$ for sandwich plate
E	Young's modulus
G_c	core shear modulus
h	core thickness
M_{ij}	bending moment tensor
n	unit outward normal to $\partial\Omega$
N_{ij}	membrane force tensor
q	lateral distributed load
Q_i	transverse shear force
s	unit tangent to the boundary $\partial\Omega$
t	plate thickness or face-sheet thickness
u_i	in-plane displacement

U $= U_N + U_M + U_Q$, strain energy density

U_N, U_M, U_Q strain energy contributed by ϵ_{ij}, θ_{ij}, and Ψ_i, respectively

V $= V_N + V_M + V_Q$, complementary energy density

V_N, V_M, V_Q complementary energy contributed to N_{ij}, M_{ij}, and Q_i, respectively

w deflection

Ψ_i shearing strain conjugated to Q_i

ϵ_{ij} stretch strain conjugated to N_{ij}

θ_{ij} bending strain conjugated to M_{ij}

ν Poisson's ratio

ϕ_i average rotation of the normal to mid-surface in i direction

$(\bar{\ })$ over a symbol denotes prescribed value

References

1 Jin, F. S. On the problem of nonlinear Reissner plate. *Acta Mech. Sinica* 1990, **22,** 748
2 Gao, Y. and Stang, G. Geometric nonlinearity: potential energy, complementary energy and the gap function, *Quart. Appl. Math.* 1989, **XLVII**(3), 487
3 Kamiya, N., Sawaki, Y., and Nakamura, Y. Finite and post buckling deflections of heated plates and shallow shell. *Boundary elements*, eds. C. A. Brebbia, T. Futagami, and M. Tanaka. Springer-Verlag, Berlin, Heidelberg, 1983, pp. 507–516
4 Tanaka, M. Large deflection analysis of thin elastic plates. *Developments in Boundary Element Methods*, Vol. 3, eds. P. K. Banerjee and S. Mukhorjee, Elsevier Applied Science Publishers, London and New York, 1984, pp. 115–136
5 Qin, Qing-Hua. Variational principles, finite element and boundary element analysis of finite deformation of plates and shells. Ph.D. Thesis, Huazhong University of Science and Technology, 1990
6 Lei, X. Y., Huang, M. K., and Wang, X. X. Geometrically nonlinear analysis of a Reissner type plate by the boundary element method. *Comput. Struct.* 1990, **37,** 911–916
7 Simpson, H. C. and Spector, S. J. On the positive of the second variation finite elasticity. *Arch. Rational Mech. Anal.* 1987, **98,** 1
8 Qin, Q. H. and Huang, Y. Y. BEM of postbuckling analysis of thin plates. *Appl. Math. Modelling* 1990, **14,** 544
9 Vander Weeen, F. Application of the boundary integral equation method to Reissner's plate model. *Int. J. Num. Meth. Eng.* 1982, **18,** 1–10
10 Pica, A., Wood, R. D., and Hinton, E. Finite element analysis of geometrically nonlinear plate behavior using a Mindlin formulation. *Comput. Struct.* 1980, **11,** 203
11 Schmit, Jr., L. A. and Manforton, G. R. Finite deflection discrete element analysis of sandwich plates and cylindrical shells with laminated faces. *AIAA J.* 1970, **8,** 1454
12 Jin, F. S. Variational principles for gravitation theory of two scale-tensor variables. *Sci. Bull. China* 1990, **35,** 1459